

# Semilocal Convergence Behavior of Halley's Method Using Kantorovich's Majorants Principle

Yonghui Ling<sup>a,\*</sup>, Xiubin Xu<sup>b,†</sup>

<sup>a</sup> *Department of Mathematics, Zhejiang University, Hangzhou 310027, China*

<sup>b</sup> *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*

**Abstract:** The present paper is concerned with the semilocal convergence problems of Halley's method for solving nonlinear operator equation in Banach space. Under some so-called majorant conditions, a new semilocal convergence analysis for Halley's method is presented. This analysis enables us to drop out the assumption of existence of a second root for the majorizing function, but still guarantee Q-cubic convergence rate. Moreover, a new error estimate based on a directional derivative of the twice derivative of the majorizing function is also obtained. This analysis also allows us to obtain two important special cases about the convergence results based on the premises of Kantorovich and Smale types.

**Keywords:** Halley's Method; Majorant Condition; Majorizing Function; Majorizing Sequence; Kantorovich-type Convergence Criterion; Smale-type Convergence Criterion

**Subject Classification:** 47J05, 65J15, 65H10.

## 1 Introduction

In this paper, we concern with the numerical approximation of the solution  $x$  of the nonlinear equation

$$F(x) = 0, \quad (1.1)$$

where  $F$  is a given nonlinear operator which maps from some nonempty open convex subset  $D$  in a Banach space  $X$  to another Banach space  $Y$ . Newton's method with initial point  $x_0$  is defined by

$$x_{k+1} = x_k - F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \dots, \quad (1.2)$$

which is the most efficient method known for solving such an equation. One of the famous results on Newton's method (1.2) is the well-known Kantorovich theorem [17], which guarantees convergence of that method to a solution using semilocal conditions. It does not require a priori existence of a solution, proving instead the existence of the solution and its uniqueness on some region. Another important result concerning Newton's method (1.2) is Smale's point estimate theory [19]. It assumes that the nonlinear operator is analytic at the initial point.

Since then, Kantorovich like theorem has been the subject of many new researches, see for example, [5, 9, 10, 25, 26, 28]. For Smale's point estimate theory, Wang and Han in [24] discussed  $\alpha$  criteria

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\*Corresponding author.

*E-mail:* lingyinghui@163.com (Y. Ling), xxu@zjnu.cn (X. Xu).

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under some weak condition and generalized this theory. In particular, Wang in [21] introduced some weak Lipschitz conditions called Lipschitz conditions with L-average, under which Kantorovich like convergence criteria and Smale's point estimate theory can be investigated together.

Recently, Ferreira and Svaiter [8] presented a new convergence analysis for Kantorovich's theorem which makes clear, with respect to Newton's method (1.2), the relationship of the majorizing function  $h$  and the nonlinear operator  $F$  under consideration. Specifically, they studied the semilocal convergence of Newton's method (1.2) under the following majorant conditions:

$$\|F'(x_0)^{-1}[F'(y) - F'(x)]\| \leq h'(\|y - x\| + \|x - x_0\|) - h'(\|x - x_0\|), \quad x, y \in \mathbf{B}(x_0, R), R > 0,$$

where  $\|y - x\| + \|x - x_0\| < R$  and  $h : [0, R) \rightarrow \mathbb{R}$  is a continuously differentiable, convex and strictly increasing function and satisfies  $h(0) > 0, h'(0) = -1$ , and has zero in  $(0, R)$ . This convergence analysis relaxes the assumptions for guaranteeing Q-quadratic convergence (see Definition 1) of Newton's method (1.2) and obtains a new estimate of the Q-quadratic convergence. This analysis was also introduced in [7] studying the local convergence of Newton's method.

Halley's method in Banach space denoted by

$$x_{k+1} = x_k - [\mathbf{I} - L_F(x_k)]^{-1}F'(x_k)^{-1}F(x_k), \quad k = 0, 1, 2, \dots, \quad (1.3)$$

where operator  $L_F(x) = \frac{1}{2}F'(x)^{-1}F''(x)F'(x)^{-1}F(x)$ , is another famous iteration for solving nonlinear equation (1.1). The results concerning convergence of this method with its modification have recently been studied under the assumptions of Newton-Kantorovich type, see for example, [1–3, 6, 12, 27]. Besides, there are also some researches concerned with Smale-type convergence for Halley's method (1.3), if the nonlinear operator  $F$  is analytic at the initial point, see for example, [11, 20, 23].

Motivated by the ideas of Ferreira and Svaiter in [8], in the rest of this paper, we study the semilocal convergence of Halley's method (1.3) under some so-called majorant conditions.

Suppose that  $F$  is a twice Fréchet differentiable operator and there exists  $x_0 \in D$  such that  $F'(x_0)$  is nonsingular. In addition, let  $R > 0$  and  $h : [0, R) \rightarrow \mathbb{R}$  be a twice continuously differentiable function. We say the operator  $F''$  satisfies the majorant conditions, if

$$\|F'(x_0)^{-1}[F''(y) - F''(x)]\| \leq h''(\|y - x\| + \|x - x_0\|) - h''(\|x - x_0\|), \quad x, y \in \mathbf{B}(x_0, R), \quad (1.4)$$

where  $\|y - x\| + \|x - x_0\| < R$  and the following assumptions hold:

$$(A1) \quad h(0) > 0, h''(0) > 0, h'(0) = -1.$$

$$(A2) \quad h'' \text{ is convex and strictly increasing in } [0, R).$$

$$(A3) \quad h \text{ has zero(s) in } (0, R). \text{ Assume that } t^* \text{ is the smallest zero and } h'(t^*) < 0.$$

Under the assumptions that the second derivative of  $F$  satisfies the majorant conditions, we establish a semilocal convergence for Halley's method (1.3). In our convergence analysis, the assumptions for guaranteeing Q-cubic convergence of Halley's method (1.3) are relaxed. In addition, we obtain a new error estimate based on a directional twice derivative of the derivative of the majorizing function. We drop out the assumption of existence of a second root for the majorizing function, still guaranteeing Q-cubic convergence. Moreover, the majorizing function even do not need to be defined beyond its first root. In particular, this convergence analysis allows us to obtain some important special cases, which includes Kantorovich-type convergence results under Lipschitz conditions and Smale-type convergence results under the  $\gamma$ -condition (see Definition 3).

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary notions and properties of the majorizing function. In Section 3, we study the majorizing function

and the results regarding only the majorizing sequence. The main results about the semilocal convergence and new error estimate are stated and proved in Section 4. In Section 5, we present two special cases of our main results. And finally in Section 6, some remarks and numerical example are offered.

## 2 Preliminaries

Let  $X$  and  $Y$  be Banach spaces. For  $x \in X$  and a positive number  $r$ , throughout the whole paper, we use  $\mathbf{B}(x, r)$  to stand for the open ball with radius  $r$  and center  $x$ , and let  $\overline{\mathbf{B}}(x, r)$  denote its closure.

Throughout this paper, for a convergent sequence  $\{x_n\}$  in  $X$ , we use the notion of Q-order of convergence (see [16] or [18] for more details).

**Definition 1.** A sequence  $\{x_n\}$  converges to  $x^*$  with Q-order (at least)  $q \geq 1$  if there exist two constants  $c \geq 0$  and  $N \geq 0$  such that for all  $n \geq N$  we have

$$\|x^* - x_{n+1}\| \leq c\|x^* - x_n\|^q.$$

For  $q = 2, 3$  the convergence is said to be (at least) Q-quadratic and Q-cubic, respectively.

The notions about Lipschitz condition (see [4, 22]) and the  $\gamma$ -condition (see [24]) are defined as follows.

**Definition 2** (Lipschitz Condition). The condition on the operator  $F$

$$\|F(x) - F(y)\| \leq L\|x - y\|, \quad x, y \in D$$

is usually called the Lipschitz condition in the domain  $D$  with constant  $L$ . If it is only required to satisfy

$$\|F(x) - F(x_0)\| \leq L\|x - x_0\|, \quad x \in \mathbf{B}(x_0, r),$$

we call it the center Lipschitz condition in the ball  $\mathbf{B}(x_0, r)$ . In particular, if  $F'(x_0)^{-1}F'$  satisfies the Lipschitz condition, i.e.

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq L\|x - y\|, \quad x, y \in \mathbf{B}(x_0, r),$$

we call it the affine covariant Lipschitz condition. The corresponding center Lipschitz condition is referred to as affine covariant center Lipschitz condition.

**Definition 3** ( $\gamma$ -Condition). Let  $F : D \subset X \rightarrow Y$  be a nonlinear operator with thrice continuously differentiable,  $D$  open and convex. Suppose  $x_0 \in D$  is a given point, and let  $0 < r \leq 1/\gamma$  be such that  $\mathbf{B}(x_0, r) \subset D$ .  $F$  is said to satisfy the  $\gamma$ -condition (with 1-order) on  $\mathbf{B}(x_0, r)$  if

$$\|F'(x_0)^{-1}F''(x)\| \leq \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}.$$

$F$  is said to satisfy the  $\gamma$ -condition with 2-order on  $\mathbf{B}(x_0, r)$  if the following relation holds:

$$\|F'(x_0)^{-1}F'''(x)\| \leq \frac{6\gamma^2}{(1 - \gamma\|x - x_0\|)^4}.$$

For the convergence analysis, we need the following useful lemmas about elementary convex analysis. The first one is slightly modified from the one in [7].

**Lemma 1.** Let  $R > 0$ . If  $g : [0, R) \rightarrow \mathbb{R}$  is continuously differentiable and convex, then

- (i)  $(1 - \theta)g'(\theta t) \leq \frac{g(t) - g(\theta t)}{t} \leq (1 - \theta)g'(t)$  for all  $t \in (0, R)$  and  $0 \leq \theta \leq 1$ .
- (ii)  $\frac{g(u) - g(\theta u)}{u} \leq \frac{g(v) - g(\theta v)}{v}$  for all  $u, v \in [0, R)$ ,  $u < v$  and  $0 \leq \theta \leq 1$ .

**Lemma 2** ([8]). Let  $I \subset \mathbb{R}$  be an interval and  $g : I \rightarrow \mathbb{R}$  be convex. Then

- (i) For any  $u_0 \in \text{int}(I)$ , there exists (in  $\mathbb{R}$ )

$$D^-g(u_0) := \lim_{u \rightarrow u_0^-} \frac{g(u_0) - g(u)}{u_0 - u} = \sup_{u < u_0} \frac{g(u_0) - g(u)}{u_0 - u}. \quad (2.1)$$

- (ii) If  $u, v, w \in I$  and  $u \leq v \leq w$ , then

$$g(v) - g(u) \leq [g(w) - g(u)] \frac{v - u}{w - u}.$$

For the convenience of analysis, we define the majorizing function with respect to Halley's method (1.3) as follows.

**Definition 4.** Let  $F : D \subset X \rightarrow Y$  be a twice continuously differentiable nonlinear operator. For a given guess  $x_0 \in D$ , we assume  $F'(x_0)$  is nonsingular. A continuously twice differentiable function  $h : [0, R) \rightarrow \mathbb{R}$  is said to be a majorizing function to  $F$  at  $x_0$ , if  $F''$  satisfies the majorant conditions in  $\mathbf{B}(x_0, R) \subset D$  and the following initial conditions:

$$\|F'(x_0)^{-1}F(x_0)\| \leq h(0), \quad \|F'(x_0)^{-1}F''(x_0)\| \leq h''(0). \quad (2.2)$$

The following lemma describes some basic properties of the majorizing function  $h$ .

**Lemma 3.** Let  $R > 0$  and let  $h : [0, R) \rightarrow \mathbb{R}$  be a twice continuously differentiable function which satisfies assumptions (A1) – (A3). Then

- (i)  $h'$  is strictly convex and strictly increasing on  $[0, R)$ .
- (ii)  $h$  is strictly convex on  $[0, R)$ ,  $h(t) > 0$  for  $t \in [0, t^*)$  and equation  $h(t) = 0$  has at most one root on  $(t^*, R)$ .
- (iii)  $-1 < h'(t) < 0$  for  $t \in (0, t^*)$ .

*Proof.* (i) follows from assumption (A2) and  $h''(0) > 0$  in (A1). (i) implies that  $h$  is strictly convex. As assumption (A1), (i) and  $h(t^*) = 0$ , we know that  $h(t) = 0$  has at most one root on  $(t^*, R)$ . Since  $h(t^*) = 0$  and  $h(0) > 0$ , one has  $h(t) > 0$  for  $t \in [0, t^*)$ . It remains to show (iii). Firstly, since  $h$  is strictly convex, we obtain from Lemma 1 that

$$h'(t) < \frac{h(t^*) - h(t)}{t^* - t}, \quad t \in [0, t^*).$$

This implies  $0 = h(t^*) > h(t) + h'(t)(t^* - t)$ . In view of  $h(t) > 0$  in  $[0, t^*)$ , we get  $h'(t) < 0$ . Secondly, as  $h'$  is strictly increasing and  $h'(0) = -1$ , we have  $h'(t) > -1$  for  $t \in (0, t^*)$ . This completes the proof.  $\square$

### 3 Halley's Method Applied to the Majorizing Function

Let

$$H_F(x) := x - [\mathbf{I} - L_F(x)]^{-1} F'(x)^{-1} F(x) \quad (3.1)$$

be the iterative function of Halley's method, where  $L_F(x) = \frac{1}{2} F'(x)^{-1} F''(x) F'(x)^{-1} F(x)$ . Suppose  $h$  is the majorizing function to  $F$  (see Definition 4). Then Halley's method applied to  $h$  can be denoted as

$$H_h(t) := t - \frac{1}{1 - L_h(t)} \cdot \frac{h(t)}{h'(t)}, \quad t \in [0, R), \quad (3.2)$$

where  $L_h(t) = h(t)h''(t)/(2h'(t)^2)$ . In order to obtain the convergence of the majorizing sequence generated by applying Halley's method to the majorizing function, we need some useful lemmas.

**Lemma 4.** *Let  $h : [0, R) \rightarrow \mathbb{R}$  be a twice continuously differentiable function and satisfy assumptions (A1) – (A3). Then we have  $0 \leq L_h(t) \leq 1/4$  for  $t \in [0, t^*]$ .*

*Proof.* Define function

$$\phi(s) = h(t) + h'(t)(s - t) + \frac{1}{2}h''(t)(s - t)^2, \quad s \in [t, t^*].$$

Then, by Lemma 3 (ii),  $\phi(t) = h(t) > 0$  for  $t \in [0, t^*)$ . In addition, we have

$$\phi(t^*) = h(t) + h'(t)(t^* - t) + \frac{1}{2}h''(t)(t^* - t)^2. \quad (3.3)$$

By using Taylor's formula, one has that

$$h(t^*) = h(t) + h'(t)(t^* - t) + \frac{1}{2}h''(t)(t^* - t)^2 + \int_0^1 (1 - \tau)[h''(t + \tau(t^* - t)) - h''(t)](t^* - t)^2 d\tau. \quad (3.4)$$

In view of  $h(t^*) = 0$  and  $h''$  is increasing, it follows from (3.3) and (3.4) that  $\phi(t^*) \leq 0$ . Thus, there exists a real root of  $\phi(s)$  in  $[t, t^*]$ . So the discriminant of  $\phi(s)$  is greater than or equal to 0, i.e.,  $h'(t)^2 - 2h''(t)h(t) \geq 0$ , which is equivalent to  $0 \leq h''(t)h(t)/h'(t)^2 \leq 1/2$ . Therefore,  $0 \leq L_h(t) \leq 1/4$  for  $t \in [0, t^*]$ . The proof is complete.  $\square$

**Lemma 5.** *Let  $h : [0, R) \rightarrow \mathbb{R}$  be a twice continuously differentiable function and satisfy assumptions (A1) – (A3). Then, for all  $t \in [0, t^*)$ ,  $t < H_h(t) < t^*$ . Moreover,  $h'(t^*) < 0$  if and only if there exists  $t \in (t^*, R)$  such that  $h(t) < 0$ .*

*Proof.* For  $t \in [0, t^*)$ , since  $h(t) > 0$ ,  $-1 < h'(t) < 0$  (from Lemma 3) and  $0 \leq L_h(t) \leq 1/4$  (from Lemma 4), one has that  $t < H_h(t)$ . Furthermore, for any  $t \in (0, t^*]$ , it follows from the definition of directional derivative (2.1) and assumption (A2) that  $D^-h''(t) > 0$ . Thus, we have

$$D^-H_h(t) = \frac{h(t)^2[3h''(t)^2 - 2h'(t)D^-h''(t)]}{[h(t)h''(t) - 2h'(t)^2]^2} > 0, \quad t \in (0, t^*].$$

This implies that  $H_h(t) < H_h(t^*) = t^*$  for any  $t \in (0, t^*)$ . So the first part of this Lemma is shown. For the second part, if  $h'(t^*) < 0$ , then it is obvious that there exists  $t \in (t^*, R)$  such that  $h(t) < 0$ . Conversely, noting that  $h(t^*) = 0$ , by Lemma 1, we have  $h(t) > h(t^*) + h'(t^*)(t - t^*)$  for  $t \in (t^*, R)$ , which implies  $h'(t^*) < 0$ . This completes the proof.  $\square$

**Remark 1.** The condition  $h'(t^*) < 0$  in (A3) implies the following properties:

- (a)  $h(t^{**}) = 0$  for some  $t^{**} \in (t^*, R)$ .

(b)  $h(t) < 0$  for some  $t \in (t^*, R)$ .

In the usual versions of Kantorovich-type and Smale-type theorems for Halley's method (e.g., [11, 12]), in order to guarantee Q-cubic convergence, condition (a) is used. As we discussed, this condition is more restrictive than condition  $h'(t^*) < 0$  in assumption (A3).

**Lemma 6.** *Let  $h : [0, R) \rightarrow \mathbb{R}$  be a twice continuously differentiable function and satisfy assumptions (A1) – (A3). Then*

$$t^* - H_h(t) \leq \left[ \frac{1}{3} \frac{h''(t^*)^2}{h'(t^*)^2} + \frac{2}{9} \frac{D^- h''(t^*)}{-h'(t^*)} \right] (t^* - t)^3, \quad t \in [0, t^*). \quad (3.5)$$

*Proof.* By the definition of  $H_h$  in (3.2), we may derive the following relation

$$\begin{aligned} t^* - H_h(t) &= \frac{1}{1 - L_h(t)} \left[ (1 - L_h(t))(t^* - t) + \frac{h(t)}{h'(t)} \right] \\ &= -\frac{1}{h'(t)(1 - L_h(t))} \int_0^1 [h''(t + \tau(t^* - t)) - h''(t)] (t^* - t)^2 (1 - \tau) d\tau \\ &\quad + \frac{t^* - t}{2((1 - L_h(t)) h'(t)^2)} \int_0^1 h''(t + \tau(t^* - t)) (t^* - t)^2 (1 - \tau) d\tau. \end{aligned}$$

Since  $h''$  is convex and  $t < t^*$ , it follows from Lemma 2 (ii) that

$$h''(t + \tau(t^* - t)) - h''(t) \leq [h''(t^*) - h''(t)] \frac{\tau(t^* - t)}{t^* - t}.$$

Then, noting that  $h''$  is strictly increasing, we have

$$t^* - H_h(t) \leq -\frac{h''(t^*) - h''(t)}{6h'(t)(1 - L_h(t))} (t^* - t)^2 + \frac{h''(t^*)h''(t)}{4h'(t)^2(1 - L_h(t))} (t^* - t)^3.$$

In view of the facts that  $h'(t) < 0$ ,  $h''(0) > 0$  and  $h', h''$  are strictly increasing on  $[0, t^*)$  by Lemma 3 and that  $0 \leq L_h(t) \leq 1/4$  for  $t \in [0, t^*]$  by Lemma 4, the preceding relation can be further reduced to

$$t^* - H_h(t) \leq \frac{2}{9} \frac{h''(t^*) - h''(t)}{-h'(t)} (t^* - t)^2 + \frac{1}{3} \frac{h''(t^*)^2}{h'(t^*)^2} (t^* - t)^3. \quad (3.6)$$

As  $h'$  is increasing,  $h'(t^*) < 0$  and  $h'(t) < 0$  in  $[0, t^*)$ , we have

$$\frac{h''(t^*) - h''(t)}{-h'(t)} \leq \frac{h''(t^*) - h''(t)}{-h'(t^*)} = \frac{1}{-h'(t^*)} \frac{h''(t^*) - h''(t)}{t^* - t} (t^* - t) \leq \frac{D^- h''(t^*)}{-h'(t^*)} (t^* - t),$$

where the last inequality follows from Lemma 2 (i). Combining the above inequality with (3.6), we conclude that (3.5) holds. This completes the proof.  $\square$

By Definition 4, if  $h$  is the majorizing function to  $F$  at  $x_0$ , then the results in Lemma 4, Lemma 5 and Lemma 6 also hold. Let  $\{t_k\}$  denote the majorizing sequence generated by

$$t_0 = 0, \quad t_{k+1} = H_h(t_k) = t_k - \frac{1}{1 - L_h(t_k)} \cdot \frac{h(t_k)}{h'(t_k)}, \quad k = 0, 1, 2, \dots \quad (3.7)$$

Therefore, by using Lemma 5 and Lemma 6, one concludes that

**Theorem 1.** *Let sequence  $\{t_k\}$  be defined by (3.7). Then  $\{t_k\}$  is well defined, strictly increasing and is contained in  $[0, t^*)$ . Moreover,  $\{t_k\}$  satisfies (3.5) and converges to  $t^*$  with Q-cubic.*

## 4 Semilocal Convergence Results for Halley's Method

In this section, we study the semilocal convergence of Halley's method (1.3) in Banach space. Assume  $F$  is a twice differentiable nonlinear operator in some convex domain  $D$ . For a given guess  $x_0 \in D$ , suppose that  $F'(x_0)^{-1}$  exists. The following lemmas, which provide clear relationships between the majorizing function and the nonlinear operator, will play key roles for the convergence analysis of Halley's method (1.3).

**Lemma 7.** *Suppose  $\|x - x_0\| \leq t < t^*$ . If  $h : [0, t^*) \rightarrow \mathbb{R}$  is twice continuously differentiable and is the majorizing function to  $F$  at  $x_0$ . Then  $F'(x)$  is nonsingular and*

$$\|F'(x)^{-1}F'(x_0)\| \leq -\frac{1}{h'(\|x - x_0\|)} \leq -\frac{1}{h'(t)}. \quad (4.1)$$

In particular,  $F'$  is nonsingular in  $\mathbf{B}(x_0, t^*)$ .

*Proof.* Take  $x \in \overline{\mathbf{B}(x_0, t)}$ ,  $0 \leq t < t^*$ . Since

$$F'(x) = F'(x_0) + \int_0^1 [F''(x_0 + \tau(x - x_0)) - F''(x_0)](x - x_0) d\tau + F''(x_0)(x - x_0),$$

by using conditions (1.4) and (2.2), we have

$$\begin{aligned} \|F'(x_0)^{-1}F'(x) - \mathbf{I}\| &\leq \int_0^1 \|F'(x_0)^{-1}[F''(x_0^\tau) - F''(x_0)]\| \|x - x_0\| d\tau + \|F'(x_0)^{-1}F''(x_0)\| \|x - x_0\| \\ &\leq \int_0^1 [h''(\tau\|x - x_0\|) - h''(0)] \|x - x_0\| d\tau + h''(0)\|x - x_0\| \\ &= h'(\|x - x_0\|) - h'(0), \end{aligned}$$

where  $x_0^\tau = x_0 + \tau(x - x_0)$ . Since  $h'(0) = -1$  and  $-1 < h'(t) < 0$  for  $(0, t^*)$  from Lemma 3, we get

$$\|F'(x_0)^{-1}F'(x) - \mathbf{I}\| \leq h'(t) - h'(0) < 1.$$

It follows from Banach lemma that  $F'(x_0)^{-1}F'(x)$  is nonsingular and (4.1) holds. The proof is complete.  $\square$

**Lemma 8.** *Suppose  $\|x - x_0\| \leq t < t^*$ . If  $h : [0, t^*) \rightarrow \mathbb{R}$  is twice continuously differentiable and is the majorizing function to  $F$  at  $x_0$ . Then  $\|F'(x_0)^{-1}F''(x)\| \leq h''(\|x - x_0\|) \leq h''(t)$ .*

*Proof.* By using (1.4), we have

$$\begin{aligned} \|F'(x_0)^{-1}F''(x)\| &\leq \|F'(x_0)^{-1}[F''(x) - F''(x_0)]\| + \|F'(x_0)^{-1}F''(x_0)\| \\ &\leq h''(\|x - x_0\|) - h''(0) + h''(0) = h''(\|x - x_0\|). \end{aligned}$$

Since  $h''$  is strictly increasing, we get  $h''(\|x - x_0\|) \leq h''(t)$ . The proof is complete.  $\square$

**Lemma 9.** *Suppose that  $h : [0, t^*) \rightarrow \mathbb{R}$  is twice continuously differentiable. Let  $\{x_k\}$  be generated by Halley's method (1.3) and  $\{t_k\}$  be generated by (3.7). If  $h$  is the majorizing function to  $F$  at  $x_0$ . Then, for all  $k = 0, 1, 2, \dots$ , we have*

- (i)  $F'(x_k)^{-1}$  exists and  $\|F'(x_k)^{-1}F'(x_0)\| \leq -1/h'(\|x_k - x_0\|) \leq -1/h'(t_k)$ .
- (ii)  $\|F'(x_0)^{-1}F''(x_k)\| \leq h''(t_k)$ .

$$(iii) \|F'(x_0)^{-1}F(x_k)\| \leq h(t_k).$$

$$(iv) [\mathbf{I} - L_F(x_k)]^{-1} \text{ exists and } \|[\mathbf{I} - L_F(x_k)]^{-1}\| \leq 1/(1 - L_h(t_k)).$$

$$(v) \|x_{k+1} - x_k\| \leq t_{k+1} - t_k.$$

*Proof.* (i)-(v) are obvious for the case  $k = 0$ . Now we assume that they hold for some  $n \in \mathbb{N}$ . By the inductive hypothesis (v) and Theorem 1, we have  $\|x_{n+1} - x_0\| \leq t_{n+1} < t^*$ . It follows from Lemma 7 and Lemma 8 that (i) and (ii) hold for  $k = n + 1$ , respectively. As for (iii), we can derive the following relation from [12]:

$$F(x_{n+1}) = \frac{1}{2}F''(x_n)L_F(x_n)(x_{n+1} - x_n)^2 + \int_0^1 (1 - \tau)[F''(x_n^\tau) - F''(x_n)](x_{n+1} - x_n)^2 d\tau,$$

where  $x_n^\tau = x_n + \tau(x_{n+1} - x_n)$ . Applying (1.4) and the inductive hypotheses (i)-(ii) and (iv)-(v), we can obtain

$$\begin{aligned} \|F'(x_0)^{-1}F(x_{n+1})\| &\leq \frac{1}{2}\|F'(x_0)^{-1}F''(x_n)\|\|L_F(x_n)\|\|x_{n+1} - x_n\|^2 \\ &\quad + \int_0^1 [h''(\tau\|x_{n+1} - x_n\| + \|x_n - x_0\|) - h''(\|x_n - x_0\|)]\|x_{n+1} - x_n\|^2(1 - \tau)d\tau \\ &\leq \frac{1}{4}h''(t_n)\frac{h(t_n)h''(t_n)}{h'(t_n)^2}(t_{n+1} - t_n)^2 + \int_0^1 [h''(\tau(t_{n+1} - t_n) + t_n) - h''(t_n)](t_{n+1} - t_n)^2(1 - \tau)d\tau \\ &= h(t_{n+1}). \end{aligned}$$

This means (iii) holds for  $k = n + 1$ . By Lemma 4 and the inductive hypotheses (i)-(iii), we get (iv) for  $k = n + 1$ . Finally, for (v), we have

$$\begin{aligned} \|x_{n+2} - x_{n+1}\| &\leq \|[\mathbf{I} - L_F(x_{n+1})]^{-1}\|\|F'(x_{n+1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_{n+1})\| \\ &\leq -\frac{1}{1 - L_h(t_{n+1})}\frac{h(t_{n+1})}{h'(t_{n+1})} = t_{n+2} - t_{n+1}. \end{aligned} \quad (4.2)$$

Therefore, the statements hold for all  $k = 0, 1, 2, \dots$ . This completes the proof.  $\square$

We are now ready to prove the semilocal convergence results (convergence, convergence rate and uniqueness) for Halley's method (1.3).

**Theorem 2.** *Let  $F : D \subset X \rightarrow Y$  be a twice continuously differentiable nonlinear operator,  $D$  open and convex. Assume that there exists a starting point  $x_0 \in D$  such that  $F'(x_0)^{-1}$  exists, and that  $h$  is the majorizing function to  $F$  at  $x_0$ , i.e., (1.4) and (2.2) hold and  $h$  satisfies assumptions (A1)–(A3). Then the sequence  $\{x_k\}$  generated by Halley's method (1.3) for solving equation (1.1) with starting point  $x_0$  is well defined, is contained in  $\mathbf{B}(x_0, t^*)$  and converges to a point  $x^* \in \mathbf{B}(x_0, t^*)$  which is the solution of equation (1.1).*

*Proof.* By Lemma 9, we conclude that the sequence  $\{x_k\}$  is well defined. By Lemma 9 (v) and Theorem 1, we have  $\|x_k - x_0\| \leq t_k < t^*$  for any  $k \in \mathbb{N}$ , which means that  $\{x_k\}$  is contained in  $\mathbf{B}(x_0, t^*)$ . It follows from (4.2) and Theorem 1 that

$$\sum_{k=N}^{\infty} \|x_{k+1} - x_k\| \leq \sum_{k=N}^{\infty} (t_{k+1} - t_k) = t^* - t_N < +\infty,$$

for any  $N \in \mathbb{N}$ . Hence  $\{x_k\}$  is a Cauchy sequence in  $\mathbf{B}(x_0, t^*)$  and so converges to some  $x^* \in \mathbf{B}(x_0, t^*)$ . The above inequality also implies that  $\|x^* - x_k\| \leq t^* - t_k$  for any  $k \in \mathbb{N}$ . It remains



to prove that  $F(x^*) = 0$ . It follows from Lemma 7 that  $\{\|F'(x_k)\|\}$  is bounded. By Lemma 9, we have

$$\|F(x_k)\| \leq \|F'(x_k)\| \|F'(x_k)^{-1} F(x_k)\| \leq \|F'(x_k)\| (1 - L_h(t_k))(t_{k+1} - t_k).$$

Letting  $k \rightarrow \infty$ , by noting the fact that  $L_h(x_k)$  is bounded (from Lemma 4) and  $\{t_k\}$  is convergent, we have  $\lim_{k \rightarrow \infty} F(x_k) = 0$ . Since  $F$  is continuous in  $\overline{\mathbf{B}(x_0, t^*)}$ ,  $\{x_k\} \subset \mathbf{B}(x_0, t^*)$  and  $\{x_k\}$  converges to  $x^*$ , we also have  $\lim_{k \rightarrow \infty} F(x_k) = F(x^*)$ . This completes the proof.  $\square$

**Theorem 3.** *Under the assumptions of Theorem 2, we have the following error bound:*

$$\|x^* - x_{k+1}\| \leq (t^* - t_{k+1}) \left( \frac{\|x^* - x_k\|}{t^* - t_k} \right)^3, \quad k = 0, 1, \dots \quad (4.3)$$

Thus, the sequence  $\{x_k\}$  generated by Halley's method (1.3) converges  $Q$ -cubic as follows

$$\|x^* - x_{k+1}\| \leq \left[ \frac{1}{3} \frac{h''(t^*)^2}{h'(t^*)^2} + \frac{2}{9} \frac{D^- h''(t^*)}{-h'(t^*)} \right] \|x^* - x_k\|^3, \quad k = 0, 1, \dots \quad (4.4)$$

*Proof.* Set  $\Gamma_F = [I - L_F(x)]^{-1}$ . Applying standard analytical techniques, one has that

$$\begin{aligned} x^* - x_{k+1} &= -\Gamma_F(x_k) F'(x_k)^{-1} [-F'(x_k)(x^* - x_k) - F(x_k)] - \Gamma_F(x_k) L_F(x_k)(x^* - x_k) \\ &= -\Gamma_F(x_k) F'(x_k)^{-1} \int_0^1 (1 - \tau) [F''(x_k^\tau) - F''(x_k)] (x^* - x_k)^2 d\tau \\ &\quad + \frac{1}{2} \Gamma_F(x_k) F'(x_k)^{-1} F''(x_k) \left[ F'(x_k)^{-1} \int_0^1 (1 - \tau) F''(x_k^\tau) (x^* - x_k)^2 d\tau \right] (x^* - x_k), \end{aligned}$$

where  $x_k^\tau = x_k + \tau(x^* - x_k)$ . Using (1.4), one has that

$$\int_0^1 \|F'(x_0)^{-1} [F''(x_k^\tau) - F''(x_k)]\| (1 - \tau) d\tau \leq \int_0^1 [h''(\tau \|x^* - x_k\| + \|x_k - x_0\|) - h''(\|x_k - x_0\|)] (1 - \tau) d\tau.$$

Then, we use Lemma 2 to obtain

$$\begin{aligned} h''(\tau \|x^* - x_k\| + \|x_k - x_0\|) - h''(\|x_k - x_0\|) &\leq h''(\tau \|x^* - x_k\| + t_k) - h''(t_k) \\ &\leq [h''(\tau(t^* - t_k) + t_k) - h''(t_k)] \frac{\|x^* - x_k\|}{t^* - t_k}. \end{aligned}$$

This together with Lemma 8 and Lemma 9, we have

$$\begin{aligned} \|x^* - x_{k+1}\| &\leq -\frac{1}{(1 - L_h(t_k))h'(t_k)} \left[ \int_0^1 [h''(\tau(t^* - t_k) + t_k) - h''(t_k)] (1 - \tau) d\tau \right] \frac{\|x^* - x_k\|^3}{t^* - t_k} \\ &\quad + \frac{1}{2} \frac{h''(t_k)}{(1 - L_h(t_k))h'(t_k)^2} \left[ \int_0^1 h''(\tau(t^* - t_k) + t_k) (1 - \tau) d\tau \right] \|x^* - x_k\|^3 \\ &= -\frac{1}{(1 - L_h(t_k))h'(t_k)} \left( \frac{\|x^* - x_k\|}{t^* - t_k} \right)^3 h(t_k) \frac{t^* - t_{k+1}}{t_{k+1} - t_k} = (t^* - t_{k+1}) \left( \frac{\|x^* - x_k\|}{t^* - t_k} \right)^3. \end{aligned}$$

This shows (4.3) holds for all  $k \in \mathbb{N}$ . (4.4) follows from Lemma 6. The proof is complete.  $\square$

**Theorem 4.** *Under the assumptions of Theorem 2, the limit  $x^*$  of the sequence  $\{x_k\}$  is the unique zero of equation (1.1) in  $\mathbf{B}(x_0, \rho)$ , where  $\rho$  is defined as  $\rho := \sup\{t \in [t^*, R) : h(t) \leq 0\}$ .*

*Proof.* We first to show the solution  $x^*$  of (1.1) is unique in  $\overline{\mathbf{B}(x_0, t^*)}$ . Assume that there exists another solution  $x^{**}$  in  $\overline{\mathbf{B}(x_0, t^*)}$ . Then  $\|x^{**} - x_0\| \leq t^*$ . Now we prove by induction that

$$\|x^{**} - x_k\| \leq t^* - t_k, \quad k = 0, 1, 2, \dots \quad (4.5)$$

It is clear that the case  $k = 0$  holds because of  $t_0 = 0$ . Assume that the above inequality holds for some  $n \in \mathbb{N}$ . By Theorem 3 we have

$$\|x^{**} - x_{k+1}\| \leq (t^* - t_{k+1}) \left( \frac{\|x^{**} - x_k\|}{t^* - t_k} \right)^3.$$

Then, by applying the inductive hypothesis (4.5) to the above inequality, one has that (4.5) also holds for  $k = n + 1$ . Since  $\{x_k\}$  converges to  $x^*$  and  $\{t_k\}$  converges to  $t^*$ , from (4.5) we conclude  $x^{**} = x^*$ . Therefore,  $x^*$  is the unique zero of (1.1) in  $\overline{\mathbf{B}(x_0, t^*)}$ .

It remains to prove that  $F$  does not have zeros in  $\mathbf{B}(x_0, \rho) \setminus \overline{\mathbf{B}(x_0, t^*)}$ . For proving this fact by contradiction, assume that  $F$  does have a zero there, i.e., there exists  $x^{**} \in D \subset X$  such that  $t^* < \|x^{**} - x_0\| < \rho$  and  $F(x^{**}) = 0$ . We will show that the above assumptions do not hold. Firstly, we have the following observation;

$$F(x^{**}) = F(x_0) + F'(x_0)(x^{**} - x_0) + \frac{1}{2}F''(x_0)(x^{**} - x_0)^2 + (1 - \tau) \int_0^1 [F''(x_0^\tau) - F''(x_0)](x^{**} - x_0)^2 d\tau, \quad (4.6)$$

where  $x_0^\tau = x_0 + \tau(x^{**} - x_0)$ . Secondly, we use (1.4) to yield

$$\begin{aligned} & \left\| (1 - \tau) \int_0^1 F'(x_0)^{-1} [F''(x_0^\tau) - F''(x_0)](x^{**} - x_0)^2 d\tau \right\| \\ & \leq \int_0^1 [h''(\tau\|x^{**} - x_0\|) - h''(0)] \|x^{**} - x_0\|^2 (1 - \tau) d\tau \\ & = h(\|x^{**} - x_0\|) - h(0) - h'(0)\|x^{**} - x_0\| - \frac{1}{2}h''(0)\|x^{**} - x_0\|^2. \end{aligned} \quad (4.7)$$

Thirdly, by applying (2.2), one has that

$$\begin{aligned} & \left\| F'(x_0)^{-1} [F(x_0) + F'(x_0)(x^{**} - x_0) + \frac{1}{2}F''(x_0)(x^{**} - x_0)^2] \right\| \\ & \geq \|x^{**} - x_0\| - \|F'(x_0)^{-1}F(x_0)\| - \frac{1}{2}\|F'(x_0)^{-1}F''(x_0)\|\|x^{**} - x_0\|^2 \\ & \geq \|x^{**} - x_0\| - h(0) - \frac{1}{2}h''(0)\|x^{**} - x_0\|^2. \end{aligned} \quad (4.8)$$

In view of  $F(x^{**}) = 0$  and  $h'(0) = -1$ , combining (4.8) and (4.7), we obtain from (4.6) that

$$h(\|x^{**} - x_0\|) - h(0) + \|x^{**} - x_0\| - \frac{1}{2}h''(0)\|x^{**} - x_0\|^2 \geq \|x^{**} - x_0\| - h(0) - \frac{1}{2}h''(0)\|x^{**} - x_0\|^2,$$

which is equivalent to  $h(\|x^{**} - x_0\|) \geq 0$ . Note that  $h$  is strictly convex by Lemma 3. Hence  $h$  is strictly positive in the interval  $(\|x^{**} - x_0\|, R)$ . So, we get  $\rho \leq \|x^{**} - x_0\|$ , which is a contradiction to the above assumptions. Therefore,  $F$  does not have zeros in  $\mathbf{B}(x_0, \rho) \setminus \overline{\mathbf{B}(x_0, t^*)}$  and  $x^*$  is the unique zero of equation (1.1) in  $\mathbf{B}(x_0, \rho)$ . The proof is complete.  $\square$

## 5 Special Cases

In this section we present two special cases of the convergence results obtained in Section 4. Namely, convergence results under an affine covariant Lipschitz condition and the  $\gamma$ -condition.

## 5.1 Convergence results under the affine covariant Lipschitz condition

In [12], by using the majorizing technique, Han and Wang studied the semilocal convergence of Halley's method (1.3) under affine covariant Lipschitz condition:

$$\|F'(x_0)^{-1}[F''(y) - F''(x)]\| \leq L\|y - x\|, \quad x, y \in D. \quad (5.1)$$

The majorizing function employed in [12] is

$$f(t) = \beta - t + \frac{\eta}{2}t^2 + \frac{L}{6}t^3. \quad (5.2)$$

If we choose this cubic polynomial as the majorizing function  $h$  in (1.4), then we can see that the majorant condition (1.4) reduced to the Lipschitz condition (5.1) and that assumptions (A1) and (A2) are satisfied for  $f$ . Moreover, if the following Kantorovich-type convergence criterion holds

$$\beta < b := \frac{2(\eta + 2\sqrt{\eta^2 + 2L})}{3(\eta + \sqrt{\eta^2 + 2L})^2}, \quad (5.3)$$

then assumption (A3) is satisfied for  $f$ . Thus, the concrete forms of Theorem 2, Theorem 3 and Theorem 4 are given as follows.

**Theorem 5.** *Let  $F : D \subset X \rightarrow Y$  be a twice continuously differentiable nonlinear operator,  $D$  open and convex. Assume that there exists a starting point  $x_0 \in D$  such that  $F'(x_0)^{-1}$  exists, and satisfies the affine covariant Lipschitz condition (5.1) and  $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$ ,  $\|F'(x_0)^{-1}F''(x_0)\| \leq \eta$ . If (5.3) holds, then the sequence  $\{x_k\}$  generated by Halley's method (1.3) for solving equation (1.1) with starting point  $x_0$  is well defined, is contained in  $\mathbf{B}(x_0, t^*)$  and converges to a point  $x^* \in \mathbf{B}(x_0, t^*)$  which is the solution of equation (1.1), where  $t^*$  is the smallest positive root of  $f$  (defined by (5.2)) in  $[0, r_1]$ , where  $r_1 = 2/(\eta + \sqrt{\eta^2 + 2L})$  is the positive root of  $f'$ . The limit  $x^*$  of the sequence  $\{x_k\}$  is the unique zero of equation (1.1) in  $\mathbf{B}(x_0, t^{**})$ , where  $t^{**}$  is the root of  $f$  in interval  $[r_1, +\infty)$ . Moreover, the following error bound holds:*

$$\|x^* - x_{k+1}\| \leq (t^* - t_{k+1}) \left( \frac{\|x^* - x_k\|}{t^* - t_k} \right)^3, \quad k = 0, 1, \dots$$

And the sequence  $\{x_k\}$   $Q$ -cubically converges as follows:

$$\|x^* - x_{k+1}\| \leq \frac{3(\eta + Lt^*)^2 + 2L(1 - \eta t^* - Lt^{*2}/2)}{9(1 - \eta t^* - Lt^{*2}/2)^2} \|x^* - x_k\|^3, \quad k = 0, 1, \dots$$

## 5.2 Convergence results under the $\gamma$ -condition

The notion of the  $\gamma$ -condition (see Definition 3) for operators in Banach spaces was introduced in [24] by Wang and Han to study Smale's point estimate theory. In this subsection, we will give the semilocal convergence results for Halley's method (1.3) under the  $\gamma$ -condition. As we will discuss, these convergence results can be applied to Smale's condition (see [19] for more details about the Smale's condition).

Smale [19] studied the convergence and error estimation of Newton's method (1.2) under the hypotheses that  $F$  is analytic and satisfies

$$\left\| F'(x_0)^{-1}F^{(n)}(x_0) \right\| \leq n!\gamma^{n-1}, \quad n \geq 2, \quad (5.4)$$

where  $x_0$  is a given point in  $D$  and  $\gamma$  is defined by

$$\gamma := \sup_{k \geq 1} \left\| \frac{F'(x_0)^{-1} F^{(k)}(x_0)}{k!} \right\|^{\frac{1}{k-1}}. \quad (5.5)$$

Wang and Han in [23] completely improved Smale's results by introducing a majorizing function

$$f(t) = \beta - t + \frac{\gamma t^2}{1 - \gamma t}, \quad \gamma > 0, \quad 0 \leq t < \frac{1}{\gamma}. \quad (5.6)$$

If we choose this function as the majorizing function  $h$ , then we can see that the majorant condition (1.4) reduces to the following condition:

$$\|F'(x_0)^{-1}[F''(y) - F''(x)]\| \leq \frac{2\gamma}{(1 - \gamma\|y - x\| - \gamma\|x - x_0\|)^3} - \frac{2\gamma}{(1 - \gamma\|x - x_0\|)^3}, \quad \gamma > 0, \quad (5.7)$$

where  $\|y - x\| + \|x - x_0\| < 1/\gamma$ , and that assumptions (A1) and (A2) are satisfied for  $f$ . Moreover, if  $\alpha := \beta\gamma < 3 - 2\sqrt{2}$ , then assumption (A3) is satisfied for  $f$ . Thus, the concrete forms of Theorem 2, Theorem 3 and Theorem 4 are given as follows.

**Theorem 6.** *Let  $F : D \subset X \rightarrow Y$  be a twice continuously differentiable nonlinear operator,  $D$  open and convex. Assume that there exists a starting point  $x_0 \in D$  such that  $F'(x_0)^{-1}$  exists, and satisfies condition (5.7),  $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$  and  $\|F'(x_0)^{-1}F''(x_0)\| \leq 2\gamma$ . If  $\alpha := \beta\gamma < 3 - 2\sqrt{2}$ , then the sequence  $\{x_k\}$  generated by Halley's method (1.3) for solving equation (1.1) with starting point  $x_0$  is well defined, is contained in  $\mathbf{B}(x_0, t^*)$  and converges to a point  $x^* \in \overline{\mathbf{B}(x_0, t^*)}$  which is the solution of equation (1.1). The limit  $x^*$  of the sequence  $\{x_k\}$  is the unique zero of equation (1.1) in  $\mathbf{B}(x_0, t^{**})$ , where  $t^*$  and  $t^{**}$  are given as*

$$t^* = \frac{1 + \alpha - \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma} \quad \text{and} \quad t^{**} = \frac{1 + \alpha + \sqrt{(1 + \alpha)^2 - 8\alpha}}{4\gamma}, \quad (5.8)$$

respectively. Moreover, the following error bound holds:

$$\|x^* - x_{k+1}\| \leq (t^* - t_{k+1}) \left( \frac{\|x^* - x_k\|}{t^* - t_k} \right)^3, \quad k = 0, 1, \dots \quad (5.9)$$

And the sequence  $\{x_k\}$   $Q$ -cubically converges as follows

$$\|x^* - x_{k+1}\| \leq \frac{8\gamma^2}{3[2(1 - \gamma t^*)^2 - 1]^2} \|x^* - x_k\|^3, \quad k = 0, 1, \dots \quad (5.10)$$

The next result gives a condition easier to be checked than condition (1.4), provided that the majorizing function  $h$  is thrice continuously differentiable.

**Lemma 10.** *Let  $F : D \subset X \rightarrow Y$  be a thrice continuously differentiable nonlinear operator,  $D$  open and convex. Let  $h : [0, R) \rightarrow \mathbb{R}$  be a thrice continuously differentiable function with convex  $h''$ . Then  $F$  satisfies condition (1.4) if and only if*

$$\|F'(x_0)^{-1}F'''(x)\| \leq h'''(\|x - x_0\|), \quad (5.11)$$

for all  $x \in D$  with  $\|x - x_0\| < R$ .

*Proof.* If  $F$  satisfies (1.4), then (5.11) holds trivially. Conversely, if  $F$  satisfies (5.11), then we have

$$\begin{aligned}\|F'(x_0)^{-1}[F''(y) - F''(x)]\| &\leq \int_0^1 \|F'(x_0)^{-1}F'''(x + \tau(y - x))\| \|y - x\| d\tau \\ &\leq \int_0^1 h'''(\|x - x_0\| + \tau\|y - x\|) \|y - x\| d\tau \\ &= h''(\|y - x\| + \|x - x_0\|) - h''(\|x - x_0\|),\end{aligned}$$

which implies that  $F$  satisfies (1.4). The proof is complete.  $\square$

If the majorizing function  $h$  is defined by (5.6), then (5.11) becomes

$$\|F'(x_0)^{-1}F'''(x)\| \leq \frac{6\gamma^2}{(1 - \gamma\|x - x_0\|)^4}, \quad (5.12)$$

which means that  $F$  satisfies the  $\gamma$ -condition with 2-order (see Definition 3) in  $\mathbf{B}(x_0, R)$ . By [24], if  $F$  satisfies the  $\gamma$ -condition with 2-order, then  $F$  satisfies the  $\gamma$ -condition (with 1-order).

One typical and important class of examples satisfying the  $\gamma$ -condition with 2-order (5.12) is the one of analytic functions. The following lemma shows that an analytic operator satisfies the  $\gamma$ -condition with 2-order.

**Lemma 11.** *Let  $F : D \rightarrow Y$  be an analytic nonlinear operator. Suppose that  $x_0 \in D$  is a given point,  $F'(x_0)$  is invertible and that  $\mathbf{B}(x_0, 1/\gamma) \subset D$ . Then  $F$  satisfies the  $\gamma$ -condition with 2-order (5.12) in  $\mathbf{B}(x_0, 1/\gamma)$ , where  $\gamma$  is defined by (5.5).*

*Proof.* For any  $x \in D$ , since  $F$  is an analytic operator, we have

$$F'(x_0)^{-1}F'''(x) = \sum_{n=0}^{\infty} \frac{1}{n!} F'(x_0)^{-1}F^{(n+3)}(x_0)(x - x_0)^n.$$

This together with (5.5) directly leads to

$$\|F'(x_0)^{-1}F'''(x)\| \leq \gamma^2 \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)(\gamma\|x - x_0\|)^n.$$

Noting that  $\gamma\|x - x_0\| < 1$  due to the assumption  $\mathbf{B}(x_0, 1/\gamma) \subset D$ , we have

$$\sum_{n=0}^{\infty} (n+3)(n+2)(n+1)(\gamma\|x - x_0\|)^n = \frac{6}{(1 - \gamma\|x - x_0\|)^4},$$

which completes the proof.  $\square$

From Lemma 10 and Lemma 11, we conclude that the semilocal convergence results obtained in Theorem 6 also hold when  $F$  is an analytic operator.

**Theorem 7.** *Let  $F : D \rightarrow F$  be an analytic operator,  $D$  open and convex. Assume that exists  $x_0 \in D$  such that  $F'(x_0)$  is nonsingular. If  $\|F'(x_0)^{-1}F(x_0)\| \leq \beta$  and  $\alpha := \beta\gamma < 3 - 2\sqrt{2}$ , where  $\gamma$  is given by (5.5). Then the sequence  $\{x_k\}$  generated by Halley's method (1.3) for solving equation (1.1) with starting point  $x_0$  is well defined, is contained in  $\mathbf{B}(x_0, t^*)$  and converges to a point  $x^* \in \overline{\mathbf{B}(x_0, t^*)}$  which is the solution of equation (1.1). The limit  $x^*$  of  $\{x_k\}$  is the unique zero of equation (1.1) in  $\mathbf{B}(x_0, t^{**})$ , where  $t^*$  and  $t^{**}$  are given in (5.8). Moreover, the error estimate and the convergence rate for  $\{x_k\}$  are characterized by (5.9) and (5.10), respectively.*

## 6 Remarks and Numerical Example

All the well-known one-point iterative methods with third-order of convergence are given by the following unified form (see [13, 15] for more details):

$$\begin{cases} x_{n+1} = x_n - H(L_F(x_n))F'(x_n)^{-1}F(x_n), \\ H(L_F(x_n)) = \mathbf{I} + \frac{1}{2}L_F(x_n) + \sum_{k \geq 2} a_k L_F(x_n)^k, \\ L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n), \quad n \in \mathbb{N}, \end{cases} \quad (6.1)$$

where  $\{a_k\}_{k \geq 2}$  is a nonnegative and nonincreasing real sequence such that

$$\sum_{k=0}^{\infty} a_k t^k < +\infty, \quad t \in [-\frac{1}{2}, \frac{1}{2}] \quad \text{with} \quad a_0 = 1, a_1 = \frac{1}{2}.$$

Thus, if  $L_F(x_n)$  exists and  $\|L_F(x_n)\| \leq 1/2$ , then (6.1) is well defined. In particular, when  $a_k = 1/2^k$  for any  $k \geq 0$ , (6.1) reduces to Halley's method (1.3).

Hernández and Romero in [15] studied the semilocal convergence of (6.1) under the following condition:

$$\|F''(x) - F''(y)\| \leq |p''(u) - p''(v)|, \quad x, y \in D, u, v \in [a, s] \text{ such that } \|x - y\| \leq |u - v|, \quad (6.2)$$

where  $p$  is a sufficiently differentiable nonincreasing and convex real function in an interval  $[a, b]$  such that  $p(a) > 0 > p(b)$  and  $p'''(t) \geq 0$  in  $[a, s]$ , and  $s$  is the unique simple solution of  $p(t) = 0$  in  $[a, b]$ .

We point out that condition (1.4) used in our convergence analysis is affine invariant but not condition (6.2) (see [4, 5] for more details about the affine invariant theory), and that the assumptions of the majorizing function used in our analysis are weaker than the ones in [15]. Furthermore, our convergence analysis provides a clear relationship between the majorizing function and the nonlinear operator, see Lemmas 7, 8 and 9.

To illustrate the theoretical results, we provide a numerical example on nonlinear Hammerstein integral equation of the second kind. Consider the integral equation:

$$u(s) = f(s) + \lambda \int_a^b k(s, t)u(t)^n dt, \quad \lambda \in \mathbb{R}, n \in \mathbb{N}, \quad (6.3)$$

where  $f$  is a given continuous function satisfying  $f(s) > 0$  for  $s \in [a, b]$  and the kernel function  $k(s, t)$  is continuous and positive in  $[a, b] \times [a, b]$ . Let  $X = Y = C[a, b]$  and  $D = \{u \in C[a, b] : u(s) \geq 0, s \in [a, b]\}$ . Then, finding a solution of (6.3) is equivalent to find a solution of  $F(x) = 0$ , where  $F : D \rightarrow C[a, b]$  is defined by

$$F(u)(s) = u(s) - f(s) - \lambda \int_a^b k(s, t)u(t)^n dt, \quad s \in [a, b], \lambda \in \mathbb{R}, n \in \mathbb{N}.$$

We adopt the max-norm. The first and second derivative of  $F$  are given by

$$F'(u)v(s) = v(s) - n\lambda \int_a^b k(s, t)u(t)^{n-1}v(t)dt, \quad v \in D,$$

and

$$F''(u)[vw](s) = -n(n-1)\lambda \int_a^b k(s, t)u(t)^{n-2}(vw)(t)dt, \quad v, w \in D.$$

Table 1: Domains of existence and uniqueness of solution for Halley's method				
$f(t) = 1$	Hernández and Romero [14]			
$\lambda$	Existence	Uniqueness	Existence	Uniqueness
0.25	$\mathbf{B}(1, 0.0346081)$	$\mathbf{B}(1, 4.06814)$	$\mathbf{B}(1, 0.0348595)$	$\mathbf{B}(1, 4.06798)$
0.5	$\mathbf{B}(1, 0.0783777)$	$\mathbf{B}(1, 2.35026)$	$\mathbf{B}(1, 0.0814400)$	$\mathbf{B}(1, 2.34809)$
0.75	$\mathbf{B}(1, 0.138260)$	$\mathbf{B}(1, 1.54454)$	$\mathbf{B}(1, 0.157580)$	$\mathbf{B}(1, 1.52953)$
1	$\mathbf{B}(1, 0.236068)$	$\mathbf{B}(1, 1)$	$\mathbf{B}(1, 0.402436)$	$\mathbf{B}(1, 0.853166)$

We choose  $[a, b] = [0, 1]$ ,  $n = 3$ ,  $x_0(t) = f(t) = 1$  and  $k(s, t)$  as the Green kernel on  $[0, 1] \times [0, 1]$  defined by

$$G(s, t) = \begin{cases} \frac{(b-s)(t-a)}{b-a} = t(1-s), & t \leq s, \\ \frac{(b-t)(s-a)}{b-a} = s(1-t), & s \leq t. \end{cases}$$

Let  $M = \max_{s \in [0, 1]} \int_0^1 |k(s, t)| dt$ . Then  $M = 1/8$ . Thus, we obtain that

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8-3|\lambda|}, \quad \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\lambda|}{8-3|\lambda|}, \quad \|F'(x_0)^{-1}F''(x_0)\| \leq \frac{6|\lambda|}{8-3|\lambda|}.$$

In addition, for any  $x, y \in D$ , we have

$$\|F'(x_0)^{-1}[F''(x) - F''(y)]\| \leq \frac{6|\lambda|}{8-3|\lambda|}\|x - y\|.$$

So, we obtain the values of  $\beta, \eta$  and  $L$  in (5.2) as follows:

$$\beta = \frac{|\lambda|}{8-3|\lambda|}, \quad \eta = \frac{6|\lambda|}{8-3|\lambda|}, \quad L = \frac{6|\lambda|}{8-3|\lambda|}.$$

Consequently, the convergence criterion (5.3) holds for any  $|\lambda| \in [0, 32/27]$ , and Theorem 5 is applicable and the sequence generated by Halley's method (1.3) with initial point  $x_0$  converges to a zero of  $F$  defined by (6.3).

For the special cases of integral equation (6.3) with  $n = 3$  when  $\lambda = 1/4, 1/2, 3/4, 1$  and  $f(t) = 1$ , the corresponding domains of existence and uniqueness of solution, together with those obtained by Hernández and Romero in [14], are given in Table 1. We notice that our convergence analysis gives better existence balls and uniqueness fields than those in [14].

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